

Bethe ansatz solutions of the τ_2 -model with arbitrary boundary fields

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Abstract

The quantum τ_2 -model with generic site-dependent inhomogeneity and arbitrary boundary fields is studied via the off-diagonal Bethe Ansatz method. The eigenvalues of the corresponding transfer matrix are given in terms of an inhomogeneous $T - Q$ relation, which is based on the operator product identities among the fused transfer matrices and the asymptotic behavior of the transfer matrices. Moreover, the associated Bethe Ansatz equations are also obtained.

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1 Introduction

The finite-size inhomogeneous τ_2 -model also known as the Baxter-Bazhanov-Stroganov model (BBS model) [1, 2, 3, 4] is a N -state spin lattice model, which is intimately related to some other integrable models under certain parameter constraints such as the chiral Potts model [5, 6, 7, 8, 9, 10] and the relativistic quantum Toda chain model [11]. Lots of papers have appeared to explain such connections and many efforts have been made to calculate the eigenvalues of the chiral Potts model by solving the τ_2 -model with a recursive functional relation [4, 12, 13]. The τ_2 -model is a simple quantum integrable models associated with cyclic representation of the Weyl algebra. Although its integrability has been proven [3] for decades, there is still no effective method to solve the model completely due to lack of a simple Q -operator solution in terms of Baxter's $T - Q$ relation. In fact, the Q -operator is a very complicated function defined in high genus space and its concrete expression is hard to be derived. Very recently, Paul Fendley had found a “parafermionic” way to diagonalise a simple solvable Hamiltonian associated with the chiral Potts model [14]. Subsequently, this method was generalised to solve the τ_2 -model with particular open boundaries [15, 16, 17]. Until very recently, the Bethe Ansatz solution of the periodic τ_2 -model with generic site-dependent inhomogeneity was obtained by constructing an inhomogeneous $T - Q$ relation with polynomial Q -functions (i.e. off-diagonal Bethe Ansatz method (ODBA) [18, 19, 20]), which provides a perspective to investigate the τ_2 -model with generic open boundary condition. Moreover, such a solution allows the authors [21] further to retrieve the corresponding Bethe states.

The aim of this paper is to explicitly construct the eigenvalues of the transfer matrix for the open τ_2 -model with the most generic inhomogeneity, where we generalise the ODBA method to solve the open inhomogeneous τ_2 -model with arbitrary integrable open boundary condition in combination with the fusion technique. By introducing an off-diagonal term in the conventional $T - Q$ relation (i.e., the inhomogeneous $T - Q$ relation), we obtain the spectrum of the generic open τ_2 -model and the associated Bethe Ansatz equations.

The outline of this paper is as follows. In Section 2, we begin with a brief introduction of the fundamental transfer matrix. In Section 3, we study the properties of the transfer matrix and employing the so-called fusion procedure [22, 23, 24] to construct the higher-spin transfer matrices, which obey an infinite fusion hierarchy. In Section 4, we obtain the

truncation identity for the fused transfer matrices when the bulk anisotropy value takes the special case $\eta = \frac{2i\pi}{p}$ and the exact functional relations of the fundamental transfer matrix. In Section 5, we give the eigenvalues of the transfer matrix in terms of some inhomogeneous $T - Q$ relation and the associated Bethe Ansatz equation. In the last Section, we summarize our results and give some discussions. Some detailed technical calculations are given in appendices A and B.

2 Transfer matrix

Let us fix an odd integer p such that $p \geq 3$, and let \mathbf{V} be a p -dimensional vector space (i.e. the local Hilbert space) with an orthonormal basis $\{|m\rangle | m \in \mathbb{Z}_p\}$. Define two $p \times p$ matrices X and Z which act on the basis as

$$X|m\rangle = q^m|m\rangle, \quad Z|m\rangle = |m+1\rangle, \quad m \in \mathbb{Z}_p, \quad (2.1)$$

where $q \equiv e^{-\eta}$ is a p -root of unity (i.e., $q^p = 1$). The embedding operators $\{X_n, Z_n | n = 1, \dots, N\}$ denote the generators of the ultra-local Weyl algebra:

$$X_n Z_m = q^{\delta_{nm}} Z_m X_n, \quad X_n^p = Z_n^p = 1, \quad \forall n, m \in \{1, \dots, N\}. \quad (2.2)$$

It has been shown that the τ_2 -model can be described by a quantum integrable spin chain [3]. In order to construct the monodromy matrix, one need to introduce the L -operators for each site of the quantum chain. The associated L -operator $L_n(u) \in \text{End}(\mathbf{C}^2 \otimes \mathbf{V})$ defined in the most general cyclic representation of $U_q(sl_2)$, is given by [3]

$$\begin{aligned} L_n(u) &= \begin{pmatrix} e^u d_n^{(+)} X_n + e^{-u} d_n^{(-)} X_n^{-1} & (g_n^{(+)} X_n^{-1} + g_n^{(-)} X_n) Z_n \\ (h_n^{(+)} X_n^{-1} + h_n^{(-)} X_n) Z_n^{-1} & e^u f_n^{(+)} X_n^{-1} + e^{-u} f_n^{(-)} X_n \end{pmatrix} \\ &= \begin{pmatrix} A_n(u) & B_n(u) \\ C_n(u) & D_n(u) \end{pmatrix}, \quad n = 1, \dots, N, \end{aligned} \quad (2.3)$$

where $d_n^{(+)}$, $d_n^{(-)}$, $g_n^{(+)}$, $g_n^{(-)}$, $h_n^{(+)}$, $h_n^{(-)}$, $f_n^{(+)}$ and $f_n^{(-)}$ are some parameters associated with each site. These parameters are subjected to two constraints,

$$g_n^{(-)} h_n^{(-)} = f_n^{(-)} d_n^{(+)}, \quad g_n^{(+)} h_n^{(+)} = f_n^{(+)} d_n^{(-)}, \quad n = 1, \dots, N, \quad (2.4)$$

which ensure that the above L -operator L_n satisfies the Yang-Baxter algebra [3],

$$R(u-v)(L_n(u) \otimes 1)(1 \otimes L_n(v)) = (1 \otimes L_n(v))(L_n(u) \otimes 1)R(u-v), \quad n = 1, \dots, N. \quad (2.5)$$

The associated R -matrix $R(u) \in \text{End}(\mathbf{C}^2 \otimes \mathbf{C}^2)$ is the well-known six-vertex R -matrix given by

$$R(u) = \begin{pmatrix} \sinh(u + \eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u + \eta) \end{pmatrix}, \quad (2.6)$$

with the crossing parameter η taking the special values

$$\eta = 2i\pi/p, \quad p = 2l + 1, \quad l = 1, 2, \dots. \quad (2.7)$$

The R -matrix satisfies the quantum Yang-Baxter equation (QYBE) [25, 26] and becomes some projectors when the spectral parameter u takes some special values as

$$\text{Antisymmetric-fusion conditions : } R(-\eta) = -2 \sinh \eta P^{(-)}, \quad (2.8)$$

$$\text{Symmetric-fusion conditions : } R(\eta) = 2 \sinh \eta \text{Diag}(\cosh \eta, 1, 1, \cosh \eta) P^{(+)}, \quad (2.9)$$

where $P^{(+)}$ ($P^{(-)}$) is the symmetric (anti-symmetric) projector of the tensor space $\mathbf{C}^2 \otimes \mathbf{C}^2$. Associated with the local L -operators $\{L_n(u) | n = 1, \dots, N\}$ given by (2.3), let us introduce the one-row monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} \mathbf{A}(u) & \mathbf{B}(u) \\ \mathbf{C}(u) & \mathbf{D}(u) \end{pmatrix} = L_N(u) L_{N-1}(u) \cdots L_1(u). \quad (2.10)$$

The local relations (2.5) imply that the monodromy matrix $T(u)$ also satisfies the Yang-Baxter algebra

$$R(u - v)(T(u) \otimes 1)(1 \otimes T(v)) = (1 \otimes T(v))(T(u) \otimes 1)R(u - v), \quad (2.11)$$

which ensures the integrability of the τ_2 -model with the periodic boundary condition [3].

Integrable open chain can be constructed as follows [27]. Let us introduce a pair of K -matrices $K^-(u)$ and $K^+(u)$. The former satisfies the reflection equation (RE) [28]

$$\begin{aligned} R_{12}(u_1 - u_2) K_1^-(u_1) R_{21}(u_1 + u_2) K_2^-(u_2) \\ = K_2^-(u_2) R_{12}(u_1 + u_2) K_1^-(u_1) R_{21}(u_1 - u_2), \end{aligned} \quad (2.12)$$

and the latter satisfies the dual RE

$$\begin{aligned} R_{12}(u_2 - u_1) K_1^+(u_1) R_{21}(-u_1 - u_2 - 2\eta) K_2^+(u_2) \\ = K_2^+(u_2) R_{12}(-u_1 - u_2 - 2\eta) K_1^+(u_1) R_{21}(u_2 - u_1). \end{aligned} \quad (2.13)$$

In order to construct the associated open spin chain, let us introduce the $\hat{L}(u)$ in the form of

$$\begin{aligned}\hat{L}_n(u) &= \begin{pmatrix} e^{-u-\eta} f_n^{(+)} X_n^{-1} + e^{u+\eta} f_n^{(-)} X_n & -(g_n^{(+)} X_n^{-1} + g_n^{(-)} X_n) Z_n \\ -(h_n^{(+)} X_n^{-1} + h_n^{(-)} X_n) Z_n^{-1} & e^{-u-\eta} d_n^{(+)} X_n + e^{u+\eta} d_n^{(-)} X_n^{-1} \end{pmatrix} \\ &= \begin{pmatrix} D_n(-u-\eta) & -B_n(-u-\eta) \\ -C_n(-u-\eta) & A_n(-u-\eta) \end{pmatrix}.\end{aligned}\quad (2.14)$$

It is easy to check that $L_n(u)$ enjoys the crossing property

$$L_n(u) = \sigma^y \hat{L}_n^t(-u-\eta) \sigma^y, \quad n = 1, \dots, N, \quad (2.15)$$

and the inverse relation

$$L_n(u) \hat{L}_n(-u) = \text{Det}_q\{L_n(u)\} \times \text{id}, \quad n = 1, \dots, N,$$

where the function $\text{Det}_q\{L_n(u)\}$ is the quantum determinant (which will be given by below (3.32)). Associated with the local L -operators $\{\hat{L}_n(u) | n = 1, \dots, N\}$ given by (2.14), let us introduce another one-row monodromy matrix $\hat{T}(u)$ (c.f., (2.10))

$$\hat{T}(u) = \begin{pmatrix} \hat{\mathbf{A}}(u) & \hat{\mathbf{B}}(u) \\ \hat{\mathbf{C}}(u) & \hat{\mathbf{D}}(u) \end{pmatrix} = \hat{L}_1(u) \hat{L}_2(u) \cdots \hat{L}_N(u). \quad (2.16)$$

For the system with integrable open boundaries, the transfer matrix $t(u)$ of the τ_2 -model with open boundaries can be constructed as [27]

$$t(u) = \text{tr}\{K^+(u)T(u)K^-(u)\hat{T}(u)\}, \quad (2.17)$$

where tr denotes trace over “auxiliary space”. The quadratic relation (2.11) and (dual) reflection equations (2.12) and (2.13) lead to the fact that the transfer matrix $t(u)$ of the τ_2 -model with different spectral parameters are mutually commutative [27], i.e., $[t(u), t(v)] = 0$, which ensures the integrability of the model by treating $t(u)$ as the generating functional of the conserved quantities.

In this paper, we consider the most generic non-diagonal $K^-(u)$ matrix found in Refs. [29, 30], which is in the form of

$$K^-(u) = \begin{pmatrix} K_{11}^-(u) & K_{12}^-(u) \\ K_{21}^-(u) & K_{22}^-(u) \end{pmatrix}, \quad (2.18)$$

with

$$\begin{aligned}
K_{11}^-(u) &= 2[\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) + \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)], \\
K_{22}^-(u) &= 2[\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) - \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)], \\
K_{12}^-(u) &= e^{\theta_-} \sinh(2u), \quad K_{21}^-(u) = e^{-\theta_-} \sinh(2u),
\end{aligned} \tag{2.19}$$

where α_- , β_- , and θ_- are three free boundary parameters. The most generic non-diagonal K -matrix $K^+(u)$ is given by

$$K^+(u) = K^-(-u - \eta)|_{(\alpha_-, \beta_-, \theta_-) \rightarrow (-\alpha_+, -\beta_+, \theta_+)}. \tag{2.20}$$

We note that the two K -matrices possess the following properties

$$K^\mp(u + i\pi) = -\sigma^z K^\mp(u) \sigma^z, \tag{2.21}$$

$$K^-(0) = \frac{1}{2} \text{tr}(K^-(0)) \times \text{id}, \quad K^-\left(\frac{i\pi}{2}\right) = \frac{1}{2} \text{tr}\left(K^-\left(\frac{i\pi}{2}\right) \sigma^z\right) \times \sigma^z, \tag{2.22}$$

where σ^α with $\alpha = x, y, z$ are the Pauli matrices.

3 Properties of the transfer matrix

3.1 Asymptotic behaviors and average values

Based on the explicit expressions (2.3) and (2.14) of the L-operators, the generic boundary matrices (2.19)-(2.20), and the definition (2.10)-(2.16) of the monodromy matrices, we note that the transfer matrix $t(u)$ given by (2.17) has the asymptotic behavior,

$$\lim_{u \rightarrow \pm\infty} t(u) = -\frac{1}{4} e^{\pm\{(2N+4)u + (N+2)\eta\}} \left\{ e^{\theta_+ - \theta_-} F^{(+)} F^{(-)} + e^{-\theta_+ + \theta_-} D^{(+)} D^{(-)} \right\} \times \text{id}, \tag{3.1}$$

where $D^{(\pm)}$ and $F^{(\pm)}$ are four constants related to the inhomogeneous parameters as follows,

$$D^{(\pm)} = \prod_{n=1}^N d_n^{(\pm)}, \quad F^{(\pm)} = \prod_{n=1}^N f_n^{(\pm)}. \tag{3.2}$$

Moreover, we can calculate the special values of the associated transfer matrix at $u = 0, \frac{i\pi}{2}$ with the help of the relations (2.22), namely,

$$t(0) = -2^3 \sinh \alpha_- \cosh \beta_- \sinh \alpha_+ \cosh \beta_+ \cosh \eta \\ \times \prod_{n=1}^N \left(e^{-\eta} d_n^{(+)} f_n^{(+)} + e^{\eta} d_n^{(-)} f_n^{(-)} - e^{\eta} g_n^{(+)} h_n^{(-)} - e^{-\eta} g_n^{(-)} h_n^{(+)} \right) \times \text{id}, \quad (3.3)$$

$$t\left(\frac{i\pi}{2}\right) = -2^3 \cosh \alpha_- \sinh \beta_- \cosh \alpha_+ \sinh \beta_+ \cosh \eta \\ \times \prod_{l=1}^N \left(e^{-\eta} d_n^{(+)} f_n^{(+)} + e^{\eta} d_n^{(-)} f_n^{(-)} + e^{\eta} g_n^{(+)} h_n^{(-)} + e^{-\eta} g_n^{(-)} h_n^{(+)} \right) \times \text{id}. \quad (3.4)$$

The expressions of the L -operators (2.3) and (2.14) allows us to derive their quasi-periodicities

$$L_n(u + i\pi) = -\sigma^z L_n(u) \sigma^z, \quad (3.5)$$

$$\hat{L}_n(u + i\pi) = -\sigma^z \hat{L}_n(u) \sigma^z. \quad (3.6)$$

The quasi-periodicity of K-matrices (2.21) enables us to obtain the associated periodicity property of the transfer matrix $t(u)$

$$t(u + i\pi) = t(u). \quad (3.7)$$

The above relation implies that the transfer matrix $t(u)$ can be expressed in terms of e^{2u} as a Laurent polynomial of the form

$$t(u) = e^{2(N+2)u} t_{N+2} + e^{2(N+1)u} t_{N+1} + \dots + e^{-2(N+2)u} t_{-(N+2)}, \quad (3.8)$$

where $\{t_n | n = N+2, N+1, \dots, -(N+2)\}$ form the $2N+5$ conserved charges. In particular, t_{N+2} and $t_{-(N+2)}$ are given by

$$t_{N+2} = -\frac{1}{4} e^{(N+2)\eta} \left\{ e^{\theta_+ - \theta_-} F^{(+)} F^{(-)} + e^{-\theta_+ + \theta_-} D^{(+)} D^{(-)} \right\}, \\ t_{-(N+2)} = -\frac{1}{4} e^{-(N+2)\eta} \left\{ e^{\theta_+ - \theta_-} F^{(+)} F^{(-)} + e^{-\theta_+ + \theta_-} D^{(+)} D^{(-)} \right\},$$

where the constants $D^{(\pm)}$ and $F^{(\pm)}$ are determined by (3.2).

Following the method in [31, 32] and using the crossing relation of the L-matrix (2.15) and the explicit expressions of the K-matrices (2.19) and (2.20), we verify that the corresponding transfer matrix $t(u)$ satisfies the following crossing relation

$$t(-u - \eta) = t(u). \quad (3.9)$$

We can define the average value $\mathcal{O}(u)$ of the matrix elements of the monodromy matrices $T(u)$ and $\hat{T}(u)$ (or the L -operators $L_n(u)$ and the \hat{L} -operators $\hat{L}_n(u)$) by using the averaging procedure [33]:

$$\mathcal{O}(u) = \prod_{m=1}^p O(u - m\eta),$$

where the operator $O(u)$ can be either $\{\mathbf{A}(u), \mathbf{B}(u), \mathbf{C}(u), \mathbf{D}(u), \hat{\mathbf{A}}(u), \hat{\mathbf{B}}(u), \hat{\mathbf{C}}(u), \hat{\mathbf{D}}(u)\}$ or $\{A_n(u), B_n(u), C_n(u), D_n(u), \hat{A}_n(u), \hat{B}_n(u), \hat{C}_n(u), \hat{D}_n(u) \mid n = 1, \dots, N\}$. It was shown in Ref. [33] that

$$\mathcal{T}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} = \mathcal{L}_N(u) \mathcal{L}_{N-1}(u) \cdots \mathcal{L}_1(u), \quad (3.10)$$

$$\hat{\mathcal{T}}(u) = \begin{pmatrix} \hat{\mathcal{A}}(u) & \hat{\mathcal{B}}(u) \\ \hat{\mathcal{C}}(u) & \hat{\mathcal{D}}(u) \end{pmatrix} = \hat{\mathcal{L}}_1(u) \hat{\mathcal{L}}_2(u) \cdots \hat{\mathcal{L}}_N(u), \quad (3.11)$$

and the average values of each L -operator and \hat{L} -operator are given by

$$\begin{aligned} \mathcal{L}_n(u) &= \begin{pmatrix} \mathcal{A}_n(u) & \mathcal{B}_n(u) \\ \mathcal{C}_n(u) & \mathcal{D}_n(u) \end{pmatrix} \\ &= \begin{pmatrix} e^{pu}\{d_n^{(+)}\}^p + e^{-pu}\{d_n^{(-)}\}^p & \{g_n^{(+)}\}^p + \{g_n^{(-)}\}^p \\ \{h_n^{(+)}\}^p + \{h_n^{(-)}\}^p & e^{pu}\{f_n^{(+)}\}^p + e^{-pu}\{f_n^{(-)}\}^p \end{pmatrix}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \hat{\mathcal{L}}_n(u) &= \begin{pmatrix} \hat{\mathcal{A}}_n(u) & \hat{\mathcal{B}}_n(u) \\ \hat{\mathcal{C}}_n(u) & \hat{\mathcal{D}}_n(u) \end{pmatrix} \\ &= \begin{pmatrix} e^{pu}\{f_n^{(-)}\}^p + e^{-pu}\{f_n^{(+)}\}^p & -\{g_n^{(+)}\}^p - \{g_n^{(-)}\}^p \\ -\{h_n^{(+)}\}^p - \{h_n^{(-)}\}^p & e^{pu}\{d_n^{(-)}\}^p + e^{-pu}\{d_n^{(+)}\}^p \end{pmatrix}, \end{aligned} \quad (3.13)$$

with $n = 1, \dots, N$. Note that the average values of the matrix elements are Laurent polynomials of e^{pu} , which implies

$$\mathcal{T}(u + \eta) = \mathcal{T}(u), \quad \mathcal{L}_n(u + \eta) = \mathcal{L}_n(u), \quad n = 1, \dots, N, \quad (3.14)$$

$$\hat{\mathcal{T}}(u + \eta) = \hat{\mathcal{T}}(u), \quad \hat{\mathcal{L}}_n(u + \eta) = \hat{\mathcal{L}}_n(u), \quad n = 1, \dots, N, \quad (3.15)$$

$$\lim_{u \rightarrow \pm\infty} \mathcal{A}(u) = e^{\pm pNu} \{D^{(\pm)}\}^p, \quad (3.16)$$

$$\lim_{u \rightarrow \pm\infty} \hat{\mathcal{A}}(u) = e^{\pm pNu} \{F^{(\mp)}\}^p, \quad (3.17)$$

$$\lim_{u \rightarrow \pm\infty} \mathcal{D}(u) = e^{\pm pNu} \{F^{(\pm)}\}^p, \quad (3.18)$$

$$\lim_{u \rightarrow \pm\infty} \hat{\mathcal{D}}(u) = e^{\pm pNu} \{D^{(\mp)}\}^p. \quad (3.19)$$

3.2 Fusion hierarchy

The main tool adopted in this paper to solve the open τ_2 -model is the so-called fusion technique, by which high-dimensional representations can be obtained from the low-dimensional ones. The fusion technique was first developed in Refs. [22, 23, 24] for R -matrices, and then generalised for K -matrices in Refs. [34, 35, 36, 37]. In recent years, this technique has been extensively used in solving lots of integrable models [38, 39]. Following the procedure in Ref. [23], we introduce the projectors

$$P_{1\dots m}^{(+)} = \frac{1}{m!} \sum_{\sigma \in S_m} \mathcal{P}_\sigma, \quad (3.20)$$

where S_m is the permutation group of m indices, and \mathcal{P}_σ is the permutation operator in the tensor space $\otimes_{k=1}^m \mathcal{C}^2$. For instance,

$$\begin{aligned} P_{12}^{(+)} &= \frac{1}{2}(1 + \mathcal{P}_{12}), \\ P_{123}^{(+)} &= \frac{1}{6}(1 + \mathcal{P}_{23}\mathcal{P}_{12} + \mathcal{P}_{12}\mathcal{P}_{23} + \mathcal{P}_{12} + \mathcal{P}_{23} + \mathcal{P}_{13}). \end{aligned}$$

The fused spin- j K^- -matrix is given by [35, 36]

$$\begin{aligned} K_{\{a\}}^{-(j)}(u) &= P_{a_1, \dots, a_{2j}}^{(+)} \prod_{k=1}^{2j} \left\{ \left[\prod_{l=1}^{k-1} R_{a_l, a_k}(2u + (k + l - 2j - 1)\eta) \right] \right. \\ &\quad \times \left. K_{a_k}^{-(\frac{1}{2})}(u + (k - j - \frac{1}{2})\eta) \right\} P_{a_1, \dots, a_{2j}}^{(+)}, \end{aligned} \quad (3.21)$$

where $\{a\} \equiv \{a_1, \dots, a_{2j}\}$ and $K^{-(\frac{1}{2})}(u) = K^-(u)$. The fused spin- j K^+ -matrix is given by duality

$$K_{\{a\}}^{+(j)}(u) = \frac{1}{f^{(j)}(u)} K_{\{a\}}^{-(j)}(-u - \eta) \Big|_{(\alpha_-, \beta_-, \theta_-) \rightarrow (-\alpha_+, -\beta_+, \theta_+)}, \quad (3.22)$$

where the normalization factor $f^{(j)}(u)$ is

$$f^{(j)}(u) = \prod_{l=1}^{2j-1} \prod_{k=1}^l [-\rho(2u + (l + k + 1 - 2j)\eta)],$$

with

$$\rho(u) = \sinh(u - \eta) \sinh(u + \eta). \quad (3.23)$$

The fused (boundary) matrices satisfy the generalized (boundary) Yang-Baxter equations [35, 36].

We introduce further the fused spin- j monodromy matrices $T_{\{a\}}^{(j)}(u)$ and $\hat{T}_{\{a\}}^{(j)}(u)$ in terms of the fundamental monodromy matrices $T^{(\frac{1}{2})}(u) = T(u)$ and $\hat{T}^{(\frac{1}{2})}(u) = \hat{T}(u)$ as follows:

$$\begin{aligned} T_{\{a\}}^{(j)}(u) &= P_{1,\dots,2j}^{(+)} T_1^{(\frac{1}{2})}(u - (j - \frac{1}{2})\eta) T_2^{(\frac{1}{2})}(u - (j - \frac{1}{2})\eta + \eta) \\ &\times \cdots T_{2j}^{(\frac{1}{2})}(u + (j - \frac{1}{2})\eta) P_{1,\dots,2j}^{(+)} \end{aligned} \quad (3.24)$$

$$\begin{aligned} \hat{T}_{\{a\}}^{(j)}(u) &= P_{1,\dots,2j}^{(+)} \hat{T}_1^{(\frac{1}{2})}(u - (j - \frac{1}{2})\eta) \hat{T}_2^{(\frac{1}{2})}(u - (j - \frac{1}{2})\eta + \eta) \\ &\times \cdots \hat{T}_{2j}^{(\frac{1}{2})}(u + (j - \frac{1}{2})\eta) P_{1,\dots,2j}^{(+)} \end{aligned} \quad (3.25)$$

The fused transfer matrices $t^{(j)}(u)$ which correspond to a spin- j auxiliary space can be constructed by the fused monodromy matrices and K -matrices as

$$t^{(j)}(u) = \text{tr}_{\{a\}} \left\{ K_{\{a\}}^{+(j)}(u) T_{\{a\}}^{(j)}(u) K_{\{a\}}^{-(j)}(u) \hat{T}_{\{a\}}^{(j)}(u) \right\}. \quad (3.26)$$

The double-row transfer matrix $t(u)$ given by (2.17) corresponds to the fundamental case $j = \frac{1}{2}$; that is $t^{(\frac{1}{2})}(u) = t(u)$. Also, the fused transfer matrices constitute commutative families

$$[t^{(j)}(u), t^{(j')}(v)] = 0, \quad j, j' \in \frac{1}{2}, 1, \frac{3}{2}, \dots \quad (3.27)$$

These transfer matrices also satisfy the so-called fusion hierarchy [34, 35, 36, 40]

$$\begin{aligned} t^{(\frac{1}{2})}(u) t^{(j-\frac{1}{2})}(u - j\eta) &= t^{(j)}(u - (j - \frac{1}{2})\eta) + \delta(u) t^{(j-1)}(u - (j + \frac{1}{2})\eta), \\ j &= \frac{1}{2}, 1, \frac{3}{2}, \dots, \end{aligned} \quad (3.28)$$

with the conventions $t^{(-\frac{1}{2})}(u) = 0$ and $t^{(0)} = \text{id}$. The coefficient $\delta(u)$, the so-called quantum determinant [22, 41, 42, 43], is given by

$$\delta(u) = -\nu(u) \text{Det}_q \{ K^+(u) \} \text{Det}_q \{ T(u) \} \text{Det}_q \{ K^-(u) \} \text{Det}_q \{ \hat{T}(u) \}, \quad (3.29)$$

where

$$\nu(u) = \frac{1}{\sinh(2u + \eta) \sinh(2u - \eta)}, \quad (3.30)$$

$$\text{Det}_q\{T(u)\} = \text{tr}_{12}\{P_{12}^-T_1(u)T_2(u+\eta)\} = \prod_{n=1}^N \text{Det}_q\{L_n(u)\}, \quad (3.31)$$

$$\text{Det}_q\{L_n(u)\} = e^{2u-\eta}d_n^{(+)}f_n^{(+)} + e^{-2u+\eta}d_n^{(-)}f_n^{(-)} - e^\eta g_n^{(+)}h_n^{(-)} - e^{-\eta}g_n^{(-)}h_n^{(+)}, \quad (3.32)$$

$$\text{Det}_q\{\hat{T}(u)\} = \text{tr}\{P_{12}^-\hat{T}_1(u)\hat{T}_2(u+\eta)\} = \prod_{n=1}^N \text{Det}_q\{\hat{L}_n(u)\}, \quad (3.33)$$

$$\text{Det}_q\{\hat{L}_n(u)\} = e^{-2u-\eta}d_n^{(+)}f_n^{(+)} + e^{2u+\eta}d_n^{(-)}f_n^{(-)} - e^\eta g_n^{(+)}h_n^{(-)} - e^{-\eta}g_n^{(-)}h_n^{(+)}, \quad (3.34)$$

$$\begin{aligned} \text{Det}_q\{K^-(u)\} &= \text{tr}_{12}\{P_{12}K_1^-(u)R_{12}(2u+\eta)K_2^-(u+\eta)\} \\ &= -2^2 \sinh(2u-2\eta) \sinh(u+\alpha_-) \sinh(u-\alpha_-) \\ &\quad \times \cosh(u+\beta_-) \cosh(u-\beta_-), \end{aligned} \quad (3.35)$$

$$\begin{aligned} \text{Det}_q\{K^+(u)\} &= \text{tr}_{12}\{P_{12}K_2^+(u+\eta)R_{12}(-2u-3\eta)K_2^+(u)\} \\ &= \text{Det}_q\{K^-(-u-2\eta)\}|_{(\alpha_-, \beta_-, \theta_-) \rightarrow (-\alpha_+, -\beta_+, \theta_+)} \\ &= 2^2 \sinh(2u+2\eta) \sinh(u+\alpha_+) \sinh(u-\alpha_+) \\ &\quad \times \cosh(u+\beta_+) \cosh(u-\beta_+). \end{aligned} \quad (3.36)$$

Let us introduce the functions $a(u)$ and $d(u)$ as follows:

$$\begin{aligned} a(u) &= -2^2 \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \sinh(u-\alpha_-) \sinh(u-\alpha_+) \\ &\quad \times \cosh(u-\beta_-) \cosh(u-\beta_+) \bar{A}(u), \end{aligned} \quad (3.37)$$

$$d(u) = a(-u-\eta), \quad (3.38)$$

where

$$\bar{A}(u) = e^{-N\eta}G^{(-)}H^{(+)} \prod_{n=1}^N \left(e^u - e^{-u+2\eta} \frac{d_n^{(-)}f_n^{(-)}}{g_n^{(-)}h_n^{(+)}} \right) \left(e^{-u} \frac{d_n^{(+)}f_n^{(+)}}{g_n^{(-)}h_n^{(+)}} - e^u \right). \quad (3.39)$$

Similar as the definitions (3.2), the constants $G^{(\pm)}$ and $H^{(\pm)}$ are

$$G^{(\pm)} = \prod_{n=1}^N g_n^{(\pm)}, \quad H^{(\pm)} = \prod_{n=1}^N h_n^{(\pm)}. \quad (3.40)$$

Then it is easy to check that the quantum determinant (3.29) can be expressed in terms of the above functions as

$$\delta(u) = a(u)d(u-\eta). \quad (3.41)$$

Similar to the τ_2 -model with the periodic condition [20], we can also use the recursive relation (3.28) and the coefficient function (3.41) to express the fused transfer matrix $t^{(j)}(u)$ in terms of the fundamental one $t^{(\frac{1}{2})}(u)$ with a $2j$ -order functional relation which can be expressed as the determinant of some $2j \times 2j$ matrix [44], namely,

$$t^{(j)}(u) = \begin{vmatrix} t(u+(j-\frac{1}{2})\eta) & -a(u+(j-\frac{1}{2})\eta) & & & \\ -d(u+(j-\frac{3}{2})\eta) & t(u+(j-\frac{3}{2})\eta) & -a(u+(j-\frac{3}{2})\eta) & & \\ & \ddots & & \ddots & \\ & & \dots & & \ddots \\ & & & -d(u-(j+\frac{1}{2})\eta) & t(u-(j+\frac{1}{2})\eta) & -a(u-(j+\frac{1}{2})\eta) \\ & & & -d(u-(j-\frac{1}{2})\eta) & t(u-(j-\frac{1}{2})\eta) \end{vmatrix},$$

$$j = \frac{1}{2}, 1, \frac{3}{2}, \dots \quad (3.42)$$

4 Truncation identity

We now proceed to formulate the desired operator identities to determine the spectrum of the transfer matrix $t(u)$ given by (2.17). For this purpose, we first derive separate truncation identities for the monodromy matrices and K -matrices. We recall that the fusion approach described in the previous section. When the crossing parameters η takes the special values $\eta = \frac{2i\pi}{p}$, one can find that the spin- $\frac{p}{2}$ fused monodromy matrices mentioned in (3.24), (3.25), all take the block-lower triangular forms [33]

$$T^{(\frac{p}{2})}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) & 0 \\ \mathcal{C}(u) & \mathcal{D}(u) & 0 \\ g(u) & h(u) & \text{Det}_q\{T(u - (\frac{p-1}{2})\eta)\} F T^{(\frac{p}{2}-1)}(u) F^{-1} \end{pmatrix}, \quad (4.1)$$

$$\hat{T}^{(\frac{p}{2})}(u) = \begin{pmatrix} \hat{\mathcal{A}}(u) & \hat{\mathcal{B}}(u) & 0 \\ \hat{\mathcal{C}}(u) & \hat{\mathcal{D}}(u) & 0 \\ \hat{g}(u) & \hat{h}(u) & \text{Det}_q\{\hat{T}(u - (\frac{p-1}{2})\eta)\} F \hat{T}^{(\frac{p}{2}-1)}(u) F^{-1} \end{pmatrix}, \quad (4.2)$$

where

$$F = M \otimes \sigma_z, \quad (4.3)$$

with

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & [2]_q & 0 & 0 & 0 \\ 0 & 0 & [3]_q & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & [\frac{p-1}{2}]_q \end{pmatrix}, \quad (4.4)$$

and

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad q = e^{-\eta}. \quad (4.5)$$

Here $\{g(u), h(u), \hat{g}(u), \hat{h}(u)\}$ are some operators, which are irrelevant to the transfer matrices. Moreover, through tedious calculations, it is found that the general non-diagonal boundary fused K^- -matrices (3.21) for $\eta = \frac{2i\pi}{p}$ and $j = \frac{p}{2}$ take the following form like (4.1)

$$K^{-(\frac{p}{2})}(u) = \mu^{(\frac{p}{2})}(u) \begin{pmatrix} \mathcal{K}_{11}^{-(\frac{p}{2})}(u) & \mathcal{K}_{12}^{-(\frac{p}{2})}(u) & 0 \\ \mathcal{K}_{21}^{-(\frac{p}{2})}(u) & \mathcal{K}_{22}^{-(\frac{p}{2})}(u) & 0 \\ k_3(u) & k_4(u) & K_{33}^{-(\frac{p}{2})}(u) \end{pmatrix}, \quad (4.6)$$

where the functions

$$\mu^{(\frac{p}{2})}(u) = \prod_{l=1}^{p-1} \prod_{k=1}^l \sinh(2u + (l+k-p+1)\eta), \quad (4.7)$$

$$\begin{aligned} \mathcal{K}_{11}^{-(\frac{p}{2})}(u) &= \sum_{l=0}^{[\frac{p}{2}]} c_p^{2l} (\sinh \alpha_- \cosh \beta_-)^{p-2l} (\cosh \alpha_- \sinh \beta_-)^{2l} \cosh(pu) \\ &+ \sum_{l=0}^{[\frac{p}{2}]} c_p^{2l+1} (\cosh \alpha_- \sinh \beta_-)^{p-2l-1} (\sinh \alpha_- \cosh \beta_-)^{2l+1} \sinh(pu) \\ &+ p(\sinh \alpha_- \cosh \beta_- \cosh(pu) + \cosh \alpha_- \sinh \beta_- \sinh(pu)), \end{aligned} \quad (4.8)$$

$$\begin{aligned} \mathcal{K}_{22}^{-(\frac{p}{2})}(u) &= \sum_{l=0}^{[\frac{p}{2}]} c_p^{2l} (\sinh \alpha_- \cosh \beta_-)^{p-2l} (\cosh \alpha_- \sinh \beta_-)^{2l} \cosh(pu) \\ &- \sum_{l=0}^{[\frac{p}{2}]} c_p^{2l+1} (\cosh \alpha_- \sinh \beta_-)^{p-2l-1} (\sinh \alpha_- \cosh \beta_-)^{2l+1} \sinh(pu) \\ &+ p(\sinh \alpha_- \cosh \beta_- \cosh(pu) - \cosh \alpha_- \sinh \beta_- \sinh(pu)), \end{aligned} \quad (4.9)$$

and the $(p-1) \times (p-1)$ matrix $K_{33}^{-(\frac{p}{2})}(u)$ is related to the spin- $\frac{p-2}{2}$ K^- -matrix by

$$K_{33}^{-(\frac{p}{2})}(u) = \frac{\text{Det}_q\{K^-(u - (\frac{p-1}{2})\eta)\}}{\sinh[2(u - (\frac{p-1}{2})\eta) + \eta]} F K^{-(\frac{p-2}{2})}(u) F^{-1}. \quad (4.10)$$

The functions $\mathcal{K}_{12}^{-(\frac{p}{2})}(u)$ and $\mathcal{K}_{21}^{-(\frac{p}{2})}(u)$ give contributions to calculating the asymptotic behavior of the eigenvalue of $t(u)$ (2.17) and possess the following forms respectively

$$\mathcal{K}_{12}^{-(\frac{p}{2})}(u) = \left(\frac{1}{2}\right)^{p-1} e^{p\theta_-} \sinh(2pu), \quad (4.11)$$

$$\mathcal{K}_{21}^{-(\frac{p}{2})}(u) = \left(\frac{1}{2}\right)^{p-1} e^{-p\theta_-} \sinh(2pu), \quad (4.12)$$

with the matrix F being given by (4.3). Moreover, $k_3(u)$ and $k_4(u)$ are two $p \times 1$ matrices which are irrelevant to the associated transfer matrix.

At the same time, the fused K^+ -matrices are given, in view of Eq. (3.22) with $\eta = \frac{2i\pi}{p}$ and $j = \frac{p}{2}$, by

$$K_{\{a\}}^{+(\frac{p}{2})}(u) = \frac{1}{f^{(\frac{p}{2})}(u)} K_{\{a\}}^{-(\frac{p}{2})}(-u - \eta) \Big|_{(\alpha_-, \beta_-, \theta_-) \rightarrow (-\alpha_+, -\beta_+, \theta_+)}. \quad (4.13)$$

The explicit expressions of the elements of the fused monodromy matrices and the K -matrices for the cases $p = 3$ are given in Appendix A.

Hence, we are finally in position to formulate the truncation identity for the fused transfer matrices $t^{(j)}(u)$ defined in (3.26). Based on the results of (4.1) and (4.2) for the fused monodromy matrices and those of (4.6) and (4.13) for the fused K -matrices, we obtain

$$t^{(\frac{p}{2})}(u) = (\tilde{\mathcal{A}}(u) + \tilde{\mathcal{D}}(u)) \times \text{id} + \delta\left(u - \left(\frac{p-1}{2}\right)\eta\right) t^{(\frac{p-2}{2})}(u), \quad (4.14)$$

where the coefficients $\tilde{\mathcal{A}}(u)$ and $\tilde{\mathcal{D}}(u)$ are given by³.

$$\begin{aligned} \tilde{\mathcal{A}}(u) &= [\mathcal{K}_{11}^{+(\frac{p}{2})}(u)\mathcal{A}(u) + \mathcal{K}_{12}^{+(\frac{p}{2})}(u)\mathcal{C}(u)][\mathcal{K}_{11}^{-(\frac{p}{2})}(u)\hat{\mathcal{A}}(u) + \mathcal{K}_{12}^{-(\frac{p}{2})}(u)\hat{\mathcal{C}}(u)] \\ &\quad + [\mathcal{K}_{11}^{+(\frac{p}{2})}(u)\mathcal{B}(u) + \mathcal{K}_{12}^{+(\frac{p}{2})}(u)\mathcal{D}(u)][\mathcal{K}_{21}^{-(\frac{p}{2})}(u)\hat{\mathcal{A}}(u) + \mathcal{K}_{22}^{-(\frac{p}{2})}(u)\hat{\mathcal{C}}(u)], \end{aligned} \quad (4.15)$$

$$\begin{aligned} \tilde{\mathcal{D}}(u) &= [\mathcal{K}_{21}^{+(\frac{p}{2})}(u)\mathcal{A}(u) + \mathcal{K}_{22}^{+(\frac{p}{2})}(u)\mathcal{C}(u)][\mathcal{K}_{11}^{-(\frac{p}{2})}(u)\hat{\mathcal{B}}(u) + \mathcal{K}_{12}^{-(\frac{p}{2})}(u)\hat{\mathcal{D}}(u)] \\ &\quad + [\mathcal{K}_{21}^{+(\frac{p}{2})}(u)\mathcal{B}(u) + \mathcal{K}_{22}^{+(\frac{p}{2})}(u)\mathcal{D}(u)][\mathcal{K}_{21}^{-(\frac{p}{2})}(u)\hat{\mathcal{B}}(u) + \mathcal{K}_{22}^{-(\frac{p}{2})}(u)\hat{\mathcal{D}}(u)]. \end{aligned} \quad (4.16)$$

³The average values $\{\mathcal{A}(u), \mathcal{B}(u), \mathcal{C}(u), \mathcal{D}(u)\}$ and $\{\hat{\mathcal{A}}(u), \hat{\mathcal{B}}(u), \hat{\mathcal{C}}(u), \hat{\mathcal{D}}(u)\}$ of the matrix elements of the monodromy matrices $T(u)$ and $\hat{T}(u)$ in (4.15) and (4.16) can be obtained from the relations in (3.10) and (3.11) which are based on the average values of each L -operator and \hat{L} -operator given in (3.12) and (3.13). For some examples of the small sites such as $N = 1, 2$, we give the explicit expressions of the average value functions and also discuss some special constraints that allow one to calculate these average value functions for an arbitrary number of the sites in Appendix B.

It is remarked that the functions $\mathcal{K}_{11}^{\pm(\frac{p}{2})}(u), \mathcal{K}_{12}^{\pm(\frac{p}{2})}(u), \mathcal{K}_{21}^{\pm(\frac{p}{2})}(u), \mathcal{K}_{22}^{\pm(\frac{p}{2})}(u)$ and the average values of each monodromy matrices are invariant under shifting with η .

Combining the fusion hierarchy (3.28) and the closing relation (4.14) for $\eta = \frac{2i\pi}{p}$, we arrive at the functional relation for the fundamental transfer matrix straightforward. Here we give an example of the functional relations for $p = 3$:

$$t(u + \eta) t(u) t(u - \eta) - \delta(u + \eta) t(u - \eta) - \delta(u) t(u + \eta) = \tilde{\mathcal{A}}(u) + \tilde{\mathcal{D}}(u) + \delta(u - \eta) t(u).$$

5 Eigenvalues of the fundamental transfer matrix

5.1 Functional relations of eigenvalues

The commutativity (3.27) of the fused transfer matrices $\{t^{(j)}(u)\}$ with different spectral parameters implies that they have common eigenstates. One can set $|\Psi\rangle$ to be a common eigenstate of these fused transfer matrices with eigenvalues $\Lambda^{(j)}(u)$, i.e.,

$$t^{(j)}(u)|\Psi\rangle = \Lambda^{(j)}(u)|\Psi\rangle.$$

The quasi-periodicity (3.7) and the cross relation (3.9) of the transfer matrix $t(u)$ implies that the corresponding eigenvalue $\Lambda(u)$ satisfies the properties

$$\Lambda(u + i\pi) = \Lambda(u), \quad (5.1)$$

$$\Lambda(-u - \eta) = \Lambda(u). \quad (5.2)$$

The asymptotic behavior (3.1) and the special values at $u = 0, \frac{i\pi}{2}$ of the transfer matrix $t(u)$ enables us to derive that the corresponding eigenvalue $\Lambda(u)$ have the following functional relations

$$\lim_{u \rightarrow \pm\infty} \Lambda(u) = -\frac{1}{4} e^{\pm\{(2N+4)u + (N+2)\eta\}} \left\{ e^{\theta_+ - \theta_-} F^{(+)} F^{(-)} + e^{-\theta_+ + \theta_-} D^{(+)} D^{(-)} \right\}, \quad (5.3)$$

$$\begin{aligned} \Lambda(0) &= -2^3 \sinh \alpha_- \cosh \beta_- \sinh \alpha_+ \cosh \beta_+ \cosh \eta \\ &\quad \times \prod_{n=1}^N \left(e^{-\eta} d_n^{(+)} f_n^{(+)} + e^{\eta} d_n^{(-)} f_n^{(-)} - e^{\eta} g_n^{(+)} h_n^{(-)} - e^{-\eta} g_n^{(-)} h_n^{(+)} \right), \end{aligned} \quad (5.4)$$

$$\begin{aligned} \Lambda\left(\frac{i\pi}{2}\right) &= -2^3 \cosh \alpha_- \sinh \beta_- \cosh \alpha_+ \sinh \beta_+ \cosh \eta \\ &\quad \times \prod_{n=1}^N \left(e^{-\eta} d_n^{(+)} f_n^{(+)} + e^{\eta} d_n^{(-)} f_n^{(-)} + e^{\eta} g_n^{(+)} h_n^{(-)} + e^{-\eta} g_n^{(-)} h_n^{(+)} \right). \end{aligned} \quad (5.5)$$

The analyticity of the L -operator (2.3), the quasi-periodicity (5.1) and the asymptotic behavior (5.3) of the eigenvalue give rise to that $\Lambda(u)$ possesses the following analytical property

$$\Lambda(u), \text{ as a function of } e^u, \text{ is a Laurent polynomial of degree } 2N + 4 \text{ like (3.8).} \quad (5.6)$$

According to the fusion hierarchy relation (3.28) and the determinant representation (3.42) of the fused transfer matrices, the eigenvalues $\Lambda^{(j)}(u)$ give some similar representation in terms of the fundamental one $\Lambda(u) = \Lambda^{(\frac{1}{2})}(u)$

$$\Lambda^{(j)}(u) = \begin{vmatrix} \Lambda(u+(j-\frac{1}{2})\eta) & -a(u+(j-\frac{1}{2})\eta) & & & \\ -d(u+(j-\frac{3}{2})\eta) & \Lambda(u+(j-\frac{3}{2})\eta) & -a(u+(j-\frac{3}{2})\eta) & & \\ & \ddots & & & \\ & & \dots & & \\ & & & \ddots & \\ & & & & -d(u-(j+\frac{1}{2})\eta) & \Lambda(u-(j+\frac{1}{2})\eta) & -a(u-(j+\frac{1}{2})\eta) \\ & & & & -d(u-(j-\frac{1}{2})\eta) & \Lambda(u-(j-\frac{1}{2})\eta) \end{vmatrix},$$

$$j = \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad (5.7)$$

where the functions $a(u)$ and $d(u)$ are given by (3.37) and (3.38). The truncation identity (4.14) of the spin- $\frac{p}{2}$ transfer matrix leads to the fact that the corresponding eigenvalue $\Lambda^{(\frac{p}{2})}(u)$ satisfies the relation

$$\Lambda^{(\frac{p}{2})}(u) = \tilde{\mathcal{A}}(u) + \tilde{\mathcal{D}}(u) + \delta\left(u - \left(\frac{p-1}{2}\right)\eta\right) \Lambda^{(\frac{p-2}{2})}(u), \quad (5.8)$$

where the functions $\tilde{\mathcal{A}}(u)$ and $\tilde{\mathcal{D}}(u)$ are given by (4.15)-(4.16). For example, the functional relation of the eigenvalue for $p = 3$ is

$$\Lambda(u + \eta) \Lambda(u) \Lambda(u - \eta) - \delta(u + \eta) \Lambda(u - \eta) - \delta(u) \Lambda(u + \eta) = \tilde{\mathcal{A}}(u) + \tilde{\mathcal{D}}(u) + \delta(u - \eta) \Lambda(u).$$

It is believed [1, 3, 45] that the relations (5.1)-(5.5), the analytic property (5.6) and the truncation identity (5.8) allow us to completely determine the eigenvalues $\Lambda(u)$ of the fundamental transfer matrix $t(u)$ given by (2.17).

5.2 T-Q relation

5.2.1 Generic case

Following the ODBA method [19] and the method developed in [20], we can express eigenvalues of $t(u)$ in terms of some inhomogeneous $T - Q$ relation

$$\Lambda(u) = a(u) \frac{Q(u - \eta)}{Q(u)} + d(u) \frac{Q(u + \eta)}{Q(u)} + 2^{(1-p)2N-4p+2} c \frac{\sinh(2u) \sinh(2u + 2\eta) F(u)}{Q(u)}, \quad (5.9)$$

where the functions $a(u)$ and $d(u)$ are given by (3.37) and (3.38), and the constant c is uniquely determined by the inhomogeneous parameters and boundary parameters,

$$\begin{aligned}
& \left(\frac{1}{2}\right)^{2p} c \left\{ e^{p(\theta_+ - \theta_-)} \{F^{(+)} F^{(-)}\}^p + e^{-p(\theta_+ - \theta_-)} \{D^{(+)} D^{(-)}\}^p - (-1)^N e^{-p(\alpha_+ + \beta_+ + \alpha_- + \beta_-)} \right. \\
& \quad \times \{G^{(-)} H^{(+)}\}^p - (-1)^N e^{p(\alpha_+ + \beta_+ + \alpha_- + \beta_-)} \{G^{(+)} H^{(-)}\}^p \left. \right\} = \frac{1}{4} \left\{ e^{\theta_+ - \theta_-} F^{(+)} F^{(-)} \right. \\
& \quad + e^{-(\theta_+ - \theta_-)} D^{(+)} D^{(-)} - (-1)^N e^{-(\alpha_+ + \beta_+ + \alpha_- + \beta_-)} \{G^{(-)} H^{(+)}\} e^{-\eta} \\
& \quad \left. - (-1)^N e^{\alpha_+ + \beta_+ + \alpha_- + \beta_-} \{G^{(+)} H^{(-)}\} e^{\eta} \right\}. \tag{5.10}
\end{aligned}$$

The trigonometric polynomial $Q(u)$ is parameterized by $(p-1)N + 2p$ Bethe roots $\{\lambda_j\}$

$$Q(u) = \prod_{j=1}^{(p-1)N+2p} \sinh(u - \lambda_j) \sinh(u + \lambda_j + \eta), \tag{5.11}$$

which will be specified by the associated BAEs (5.16) below. The function $F(u)$ which is a Laurent polynomial of degree $p(2N+4)$ is given by

$$F(u) = \tilde{\mathcal{A}}(u) + \tilde{\mathcal{D}}(u) - \bar{\mathcal{A}}(u) - \bar{\mathcal{D}}(u), \tag{5.12}$$

$$\bar{\mathcal{A}}(u) = \prod_{m=1}^p a(u - m\eta), \quad \bar{\mathcal{D}}(u) = \prod_{m=1}^p d(u - m\eta), \tag{5.13}$$

where the functions $\tilde{\mathcal{A}}(u)$ and $\tilde{\mathcal{D}}(u)$ are given in (4.15) and (4.16) respectively.

According to the relations (3.14)-(3.15), the definitions (5.13), (4.15) and (4.16) and the explicit expression (4.8)-(4.13) of the elements of the K -matrices, the function $F(u)$ can be reduced as a Laurent polynomial of e^{pu} with a degree $2N+4$ (i.e. there are only $2N+5$ non-vanishing coefficients), namely,

$$F(u) = \sum_{l=0}^{2N+4} F^{(2p(N+2-l))} (\{d_n^\pm, f_n^\pm, g_n^\pm, h_n^\pm, d_n^\pm, \alpha_\pm, \beta_\pm, \theta_\pm\}) e^{p(2N+4-2l)u}, \tag{5.14}$$

where the $2N+5$ coefficients $\{F^{(2p(N+2-l))} | l = 0, 1, \dots, 2N+4\}$ are polynomial of the inhomogeneity parameters $\{d_n^\pm, f_n^\pm, g_n^\pm, h_n^\pm | n = 1, 2, \dots, N\}$ and the boundary parameters $\{\alpha_\pm, \beta_\pm, \theta_\pm\}$, and that we can easily prove that the function $F(u)$ holds the crossing property

$$F(-u - \eta) = F(u). \tag{5.15}$$

The fact that the constant c satisfies the relation (5.10) ensures that $\Lambda(u)$ given by (5.9) matches the asymptotic behavior (5.3). The $(p-1)N + 2p$ parameters $\{\lambda_j | j = 1, \dots, (p-$

$1)N + 2p\}$ satisfy the associated Bethe Ansatz equations (BAEs)

$$a(\lambda_j)Q(\lambda_j - \eta) + d(\lambda_j)Q(\lambda_j + \eta) + 2^{2(1-p)N-4p+2}c \sinh(2\lambda_j) \\ \times \sinh(2\lambda_j + 2\eta)F(\lambda_j) = 0, \quad j = 1, \dots, (p-1)N + 2p, \quad (5.16)$$

which assure that $\Lambda(u)$ given by (5.9) is indeed a trigonometric polynomial of u . It is easy to check that $\Lambda(u)$ given by the inhomogeneous $T-Q$ relation (5.9) satisfies the properties (5.1)-(5.2) and the functional relations (5.3)-(5.5). Using the method in the appendix A of [20], we have checked that the $T-Q$ relation (5.9) also make the functional relation (5.8) fulfilled. Therefore, the resulting expression of $\Lambda(u)$ constructed by the inhomogeneous $T-Q$ relation (5.9) is the eigenvalue of the fundamental transfer matrix $t(u)$ of the τ_2 -model with generic boundary condition.

5.2.2 Degenerate case

The third term of the $T-Q$ relation (5.9) does not vanish when the generic inhomogeneous parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$ (only obey the constraint (2.4) which ensure the integrability of the model) and the boundary parameters are absence of restriction. Here we consider some special case making the inhomogeneous term vanishes. When the inhomogeneous parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$ and the boundary parameters $\{\alpha_{\pm}, \beta_{\pm}, \theta_{\pm}\}$ obey the following extra constraints besides (2.4):

$$e^{\theta_+ - \theta_-} F^{(+)} F^{(-)} + e^{-(\theta_+ - \theta_-)} D^{(+)} D^{(-)} - (-1)^N e^{-(\alpha_+ + \beta_+ + \alpha_- + \beta_-)} \{G^{(-)} H^{(+)}\} e^{-\eta} \\ - (-1)^N e^{\alpha_+ + \beta_+ + \alpha_- + \beta_-} \{G^{(+)} H^{(-)}\} e^{\eta} = 0, \quad (5.17)$$

$$F^{(2(N+2-l)p)}(\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)}, \alpha_{\pm}, \beta_{\pm}, \theta_{\pm}\}) = 0, \quad l = 1, \dots, N+2, \quad (5.18)$$

where $D^{(\pm)}, F^{(\pm)}, G^{(\pm)}$ and $H^{(\pm)}$ are given by (3.2) and (3.40), and each $F^{(2(N+2-l)p)}$ given in (5.14). The corresponding inhomogeneous $T-Q$ relation (5.9) reduces to the conventional one [25]:

$$\Lambda(u) = a(u) \frac{\bar{Q}(u - \eta)}{\bar{Q}(u)} + d(u) \frac{\bar{Q}(u + \eta)}{\bar{Q}(u)}, \quad (5.19)$$

where the function $\bar{Q}(u)$ becomes [18, 19, 40, 46, 47, 48, 49]

$$\bar{Q}(u) = \prod_{j=1}^M \sinh(u - \lambda_j) \sinh(u + \lambda_j + \eta). \quad (5.20)$$

Here the positive integer M has to satisfy the following constraint in order to match the asymptotic behavior (5.3) of $\Lambda(u)$, namely,

$$\begin{aligned} & \{e^{\theta_+ - \theta_-} F^{(+)} F^{(-)} + e^{-(\theta_+ - \theta_-)} D^{(+)} D^{(-)}\} - (-1)^N e^{-(\alpha_+ + \beta_+ + \alpha_- + \beta_-)} G^{(-)} H^{(+)} e^{-(2N+2M+1)\eta} \\ & - (-1)^N e^{(\alpha_+ + \beta_+ + \alpha_- + \beta_-)} G^{(+)} H^{(-)} e^{(2N+2M+1)\eta} = 0. \end{aligned} \quad (5.21)$$

Moreover, the M parameters $\{\lambda_j | j = 1, \dots, M\}$ need to satisfy the associated BAEs

$$\frac{a(\lambda_j)}{d(\lambda_j)} = -\frac{\bar{Q}(\lambda_j + \eta)}{\bar{Q}(\lambda_j - \eta)}, \quad j = 1, \dots, M. \quad (5.22)$$

It is easy to check that the relation (5.17) give rise to $F^{(p(2N+4))} = F^{(-p(2N+4))} = 0$. Together with (5.18) and the constrained case in (5.17), we get that the function $F(u)$ indeed vanishes, namely, $F(u) = 0$. Substituting (5.19) into (5.7), we have

$$\begin{aligned} \Lambda^{(\frac{p}{2})}(u) &= \bar{A}(u) + \bar{D}(u) + \delta(u - (\frac{p-1}{2})\eta) \Lambda^{(\frac{p-2}{2})}(u) \\ &= \tilde{A}(u) + \tilde{D}(u) + \delta(u - (\frac{p-1}{2})\eta) \Lambda^{(\frac{p-2}{2})}(u). \end{aligned} \quad (5.23)$$

Similar with the periodic case, we can also prove that the reduced $T - Q$ relation (5.19) satisfies the functional relations (5.3)-(5.5) of the transfer matrix and the truncation identity of the fused transfer matrices (5.8) when the inhomogeneity parameters and the boundary parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)}, \alpha_{\pm}, \beta_{\pm}, \theta_{\pm} | n = 1, \dots, N\}$ satisfy the constraints (2.4), (5.17), (5.18) and (5.21).

6 Conclusion

In this paper, we have studied the most general cyclic representation of the quantum τ_2 -model (also known as Baxter-Bazhanov-Stroganov (BBS) model) with generic integrable boundary conditions via the ODBA method [19]. Based on the truncation identity (4.14) of the fused transfer matrices obtained from the fusion technique, we construct the corresponding inhomogeneous $T - Q$ relation (5.9) and the associated BAEs (5.16) for the eigenvalue of the fundamental transfer matrix $t(u)$.

It is remarked that if the generic inhomogeneity parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$ (only obey the constraint (2.4) which ensures the integrability of the model) and the boundary parameters take the generic values, the inhomogeneous term (i.e., the third

term) in the $T - Q$ relation (5.9) *does not* vanish, as long as one requires a polynomial Q -function. However, if these inhomogeneity parameters and the boundary parameters satisfy the extra constraints (5.17), (5.18) and (5.21), the resulting inhomogeneous $T - Q$ relation (5.9) reduces to the conventional one (5.19).

Note added: After this paper was completed we became aware of the recent results reported in [50]. The authors use the Sklyanin's separation of variables (SoV) method [51, 52] to study the spectral problem for the open τ_2 -model with some constraints on inhomogeneous parameters and also on the boundary parameters.

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A Specific cases of the fused K -matrices

In this appendix we present the explicit expressions of the matrix elements of the fused K -matrices $K^{\pm(\frac{p}{2})}(u)$ given in (4.6) and (4.13) for the $p = 3$ case as an example. In this case, the corresponding matrix elements are

$$\begin{aligned} \mathcal{K}_{11}^{-\left(\frac{3}{2}\right)}(u) &= \left(\frac{1}{2}\right)^2 \left\{ (\sinh \alpha_- \cosh \beta_-)^3 \cosh(3u) + 3(\sinh \alpha_- \cosh \beta_-)^2 \cosh \alpha_- \sinh \beta_- \sinh(3u) \right. \\ &\quad + 3 \sinh \alpha_- \cosh \beta_- (\cosh \alpha_- \sinh \beta_-)^2 \cosh(3u) + (\cosh \alpha_- \sinh \beta_-)^3 \sinh(3u) \\ &\quad \left. + 3 \sinh \alpha_- \cosh \beta_- \cosh(3u) + 3 \cosh \alpha_- \sinh \beta_- \sinh(3u) \right\}, \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} \mathcal{K}_{22}^{-\left(\frac{3}{2}\right)}(u) &= \left(\frac{1}{2}\right)^2 \left\{ (\sinh \alpha_- \cosh \beta_-)^3 \cosh(3u) - 3(\sinh \alpha_- \cosh \beta_-)^2 \cosh \alpha_- \sinh \beta_- \sinh(3u) \right. \\ &\quad + 3 \sinh \alpha_- \cosh \beta_- (\cosh \alpha_- \sinh \beta_-)^2 \cosh(3u) - (\cosh \alpha_- \sinh \beta_-)^3 \sinh(3u) \\ &\quad \left. + 3 \sinh \alpha_- \cosh \beta_- \cosh(3u) - 3 \cosh \alpha_- \sinh \beta_- \sinh(3u) \right\}, \end{aligned} \quad (\text{A.2})$$

$$\mathcal{K}_{12}^{-(\frac{3}{2})}(u) = \left(\frac{1}{2}\right)^2 e^{3\theta_-} \sinh(6u), \quad \mathcal{K}_{21}^{-(\frac{3}{2})}(u) = \left(\frac{1}{2}\right)^2 e^{-3\theta_-} \sinh(6u), \quad (\text{A.3})$$

$$K_{33}^{-(\frac{3}{2})}(u) = \frac{\text{Det}_q\{K^-(u-\eta)\}}{\sinh(2u-\eta)} \sigma^z K^-(u) \sigma^z, \quad (\text{A.4})$$

and

$$\begin{aligned} \mathcal{K}_{11}^{+(\frac{3}{2})}(u) &= \left(\frac{1}{2}\right)^2 \left\{ -(\sinh \alpha_+ \cosh \beta_+)^3 \cosh(3u) + 3(\sinh \alpha_+ \cosh \beta_+)^2 \cosh \alpha_+ \sinh \beta_+ \sinh(3u) \right. \\ &\quad - 3 \sinh \alpha_+ \cosh \beta_+ (\cosh \alpha_+ \sinh \beta_+)^2 \cosh(3u) + (\cosh \alpha_+ \sinh \beta_+)^3 \sinh(3u) \\ &\quad \left. - 3 \sinh \alpha_+ \cosh \beta_+ \cosh(3u) + 3 \cosh \alpha_+ \sinh \beta_+ \sinh(3u) \right\}, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \mathcal{K}_{22}^{+(\frac{3}{2})}(u) &= -\left(\frac{1}{2}\right)^2 \left\{ (\sinh \alpha_+ \cosh \beta_+)^3 \cosh(3u) + 3(\sinh \alpha_+ \cosh \beta_+)^2 \cosh \alpha_+ \sinh \beta_+ \sinh(3u) \right. \\ &\quad + 3 \sinh \alpha_+ \cosh \beta_+ (\cosh \alpha_+ \sinh \beta_+)^2 \cosh(3u) + (\cosh \alpha_+ \sinh \beta_+)^3 \sinh(3u) \\ &\quad \left. + 3 \sinh \alpha_+ \cosh \beta_+ \cosh(3u) + 3 \cosh \alpha_+ \sinh \beta_+ \sinh(3u) \right\}, \end{aligned} \quad (\text{A.6})$$

$$\mathcal{K}_{12}^{+(\frac{3}{2})}(u) = -\left(\frac{1}{2}\right)^2 e^{3\theta_+} \sinh(6u), \quad \mathcal{K}_{21}^{+(\frac{3}{2})}(u) = -\left(\frac{1}{2}\right)^2 e^{-3\theta_+} \sinh(6u), \quad (\text{A.7})$$

$$K_{33}^{+(\frac{3}{2})}(u) = -\frac{\text{Det}_q\{K^+(u-\eta)\}}{\sinh 2u} \sigma^z K^+(u) \sigma^z. \quad (\text{A.8})$$

B Explicit expression of the average value functions

In this appendix we discuss certain properties of the average values of the matrix elements of the monodromy matrices $T(u)$ and $\hat{T}(u)$ given by (3.10) and (3.11) respectively. Here we present the explicit expressions of these average value functions $\mathcal{A}(u), \mathcal{B}(u), \mathcal{C}(u), \mathcal{D}(u)$ and $\hat{\mathcal{A}}(u), \hat{\mathcal{B}}(u), \hat{\mathcal{C}}(u), \hat{\mathcal{D}}(u)$ for some small sites cases (namely, $N = 1, 2$). For $N = 1$, they are given by

$$\mathcal{A}(u) = e^{pu} \{d_1^{(+)}\}^p + e^{-pu} \{d_1^{(-)}\}^p, \quad (\text{B.1})$$

$$\mathcal{D}(u) = e^{pu} \{f_1^{(+)}\}^p + e^{-pu} \{f_1^{(-)}\}^p, \quad (\text{B.2})$$

$$\mathcal{B}(u) = \{g_1^{(+)}\}^p + \{g_1^{(-)}\}^p, \quad \mathcal{C}(u) = \{h_1^{(+)}\}^p + \{h_1^{(-)}\}^p, \quad (\text{B.3})$$

$$\hat{\mathcal{A}}(u) = e^{pu}\{f_1^{(-)}\}^p + e^{-pu}\{f_1^{(+)}\}^p, \quad (\text{B.4})$$

$$\hat{\mathcal{D}}(u) = e^{pu}\{d_1^{(-)}\}^p + e^{-pu}\{d_1^{(+)}\}^p, \quad (\text{B.5})$$

$$\hat{\mathcal{B}}(u) = -\{g_1^{(+)}\}^p - \{g_1^{(-)}\}^p, \quad \hat{\mathcal{C}}(u) = -\{h_1^{(+)}\}^p - \{h_1^{(-)}\}^p. \quad (\text{B.6})$$

For $N = 2$, they are

$$\begin{aligned} \mathcal{A}(u) &= e^{2pu}\{d_1^{(+)}d_2^{(+)}\}^p + e^{-2pu}\{d_1^{(-)}d_2^{(-)}\}^p + \{d_1^{(-)}d_2^{(+)}\}^p + \{d_1^{(+)}d_2^{(-)}\}^p \\ &+ \left(\{g_2^{(+)}\}^p + \{g_2^{(-)}\}^p\right)\left(\{h_1^{(+)}\}^p + \{h_1^{(-)}\}^p\right), \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} \mathcal{D}(u) &= e^{2pu}\{f_1^{(+)}f_2^{(+)}\}^p + e^{-2pu}\{f_1^{(-)}f_2^{(-)}\}^p + \{f_1^{(-)}f_2^{(+)}\}^p + \{f_1^{(+)}f_2^{(-)}\}^p \\ &+ \left(\{g_1^{(+)}\}^p + \{g_1^{(-)}\}^p\right)\left(\{h_2^{(+)}\}^p + \{h_2^{(-)}\}^p\right), \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} \mathcal{B}(u) &= e^{pu}\left\{\left(\{g_1^{(+)}\}^p + \{g_1^{(-)}\}^p\right)\{d_2^{(+)}\}^p + \left(\{g_2^{(+)}\}^p + \{g_2^{(-)}\}^p\right)\{f_1^{(+)}\}^p\right\} \\ &+ e^{-pu}\left\{\left(\{g_1^{(+)}\}^p + \{g_1^{(-)}\}^p\right)\{d_2^{(-)}\}^p + \left(\{g_2^{(+)}\}^p + \{g_2^{(-)}\}^p\right)\{f_1^{(-)}\}^p\right\}, \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} \mathcal{C}(u) &= e^{pu}\left\{\left(\{h_1^{(+)}\}^p + \{h_1^{(-)}\}^p\right)\{f_2^{(+)}\}^p + \left(\{h_2^{(+)}\}^p + \{h_2^{(-)}\}^p\right)\{d_1^{(+)}\}^p\right\} \\ &+ e^{-pu}\left\{\left(\{h_1^{(+)}\}^p + \{h_1^{(-)}\}^p\right)\{f_2^{(-)}\}^p + \left(\{h_2^{(+)}\}^p + \{h_2^{(-)}\}^p\right)\{d_1^{(-)}\}^p\right\}, \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} \hat{\mathcal{A}}(u) &= e^{2pu}\{f_1^{(-)}f_2^{(-)}\}^p + e^{-2pu}\{f_1^{(+)}f_2^{(+)}\}^p + \{f_1^{(-)}f_2^{(+)}\}^p + \{f_1^{(+)}f_2^{(-)}\}^p \\ &+ \left(\{g_1^{(+)}\}^p + \{g_1^{(-)}\}^p\right)\left(\{h_2^{(+)}\}^p + \{h_2^{(-)}\}^p\right), \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} \hat{\mathcal{D}}(u) &= e^{2pu}\{d_1^{(-)}d_2^{(-)}\}^p + e^{-2pu}\{d_1^{(+)}d_2^{(+)}\}^p + \{d_1^{(-)}d_2^{(+)}\}^p + \{d_1^{(+)}d_2^{(-)}\}^p \\ &+ \left(\{g_2^{(+)}\}^p + \{g_2^{(-)}\}^p\right)\left(\{h_1^{(+)}\}^p + \{h_1^{(-)}\}^p\right), \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} \hat{\mathcal{B}}(u) &= -e^{pu}\left\{\left(\{g_1^{(+)}\}^p + \{g_1^{(-)}\}^p\right)\{d_2^{(-)}\}^p + \left(\{g_2^{(+)}\}^p + \{g_2^{(-)}\}^p\right)\{f_1^{(-)}\}^p\right\} \\ &- e^{-pu}\left\{\left(\{g_1^{(+)}\}^p + \{g_1^{(-)}\}^p\right)\{d_2^{(+)}\}^p + \left(\{g_2^{(+)}\}^p + \{g_2^{(-)}\}^p\right)\{f_1^{(+)}\}^p\right\}, \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} \hat{\mathcal{C}}(u) &= -e^{pu}\left\{\left(\{h_1^{(+)}\}^p + \{h_1^{(-)}\}^p\right)\{f_2^{(-)}\}^p + \left(\{h_2^{(+)}\}^p + \{h_2^{(-)}\}^p\right)\{d_1^{(-)}\}^p\right\} \\ &- e^{-pu}\left\{\left(\{h_1^{(+)}\}^p + \{h_1^{(-)}\}^p\right)\{f_2^{(+)}\}^p + \left(\{h_2^{(+)}\}^p + \{h_2^{(-)}\}^p\right)\{d_1^{(+)}\}^p\right\}. \end{aligned} \quad (\text{B.14})$$

The results show that the average value functions become tedious when the number of the lattice sites N goes large. However, they can be worked out for an arbitrary N when the

inhomogeneous parameters $\{g_n^{(+)}, g_n^{(-)}, h_n^{(+)}, h_n^{(-)}\}$ associated with each site n satisfy some constraints. One case is the parameters $\{g_n^{(+)}, g_n^{(-)}, h_n^{(+)}, h_n^{(-)}|n = 1, \dots, N\}$ obey the chiral Potts constraints [33]

$$\frac{\{g_n^{(+)}\}^p + \{g_n^{(-)}\}^p}{\{h_n^{(+)}\}^p + \{h_n^{(-)}\}^p} = \lambda, \quad (\text{B.15})$$

where λ is arbitrary constant. The other case is that the parameters $d_n^{(+)}, d_n^{(-)}, g_n^{(+)}, g_n^{(-)}, h_n^{(+)}, h_n^{(-)}, f_n^{(+)}$ and $f_n^{(-)}$ are independent of the site (i.e., n), which corresponds to the homogeneous model. In the both cases, the average values of L -operators $\{\mathcal{L}_n(u)|n = 1, 2, \dots, N\}$ given in (3.12) (or $\{\hat{\mathcal{L}}_n(u)|n = 1, 2, \dots, N\}$ given in (3.13)) with different sites commute with each other. Thus one can diagonalize them simultaneously. Then the average values of the elements of monodromy matrices (3.10) and (3.11) can be easy to obtain.

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